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A p -ADIC UPPER BOUND FOR THE COEFFICIENTS OF A POWER SERIES SATISFYING AN ALGEBRAIC DIFFERENTIAL EQUATION

ขอบเขตบน p -แอดิกสำหรับสัมประสิทธิ์ของอนุกรมกำลัง
ที่คล้อยตามสมการเชิงอนุพันธ์พีชคณิต

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ABSTRACT

Using a method of K. Mahler, a p -adic upper bound for the coefficients of a power series satisfying an algebraic differential equation is established.

บทคัดย่อ

งานวิจัยนี้เป็นการพิสูจน์ โดยใช้วิธีของ เค มาห์เลอร์ ในการหาขอบเขตบน p -แอดิกสำหรับสัมประสิทธิ์ของอนุกรมกำลังที่เป็นผลเฉลยของสมการเชิงอนุพันธ์พีชคณิต

INTRODUCTION

Let $f(z) = \sum_{h=0}^{\infty} f_h z^h$ be a power series, with algebraic coefficients, which satisfies an algebraic differential equation. Popken⁷ proved a remarkable result that either $f_h = 0$ or there exists a positive constant c such that for all sufficiently large suffixes h ,

$$|f_h| \geq \exp(-ch(\log h)^2)$$

Mahler⁵ proved a p -adic analogue of this theorem in 1973. These results have been found useful in constructing a large class of transcendently transcendental functions. The case of lower bound then seems to be settled. As for the case of upper bound, Mahler³ proved in the classical case that for f_h being complex numbers, there exist two positive constants γ_1 and γ_2 such that for all $h = 0, 1, \dots$

$$|f_h| \leq \gamma_1 (h!)^{\gamma_2}$$

The p -adic analogue of this last inequality does not seem to be explicitly established although as mentioned in Sibuya and Sperber⁹, it is implicit in Mahler⁶. It is our objective here to make explicit the proof of this p -adic analogue.

MATERIALS AND METHODS

The proof follows very closely the line of Mahler⁶. Sibuya and Sperber in the paper just cited have somewhat sharpened these results using some heavy machinery from differential equations. We feel that the following proof is simpler and merits one's attention.

RESULTS

Theorem. If

$$f(z) = \sum_{h=0}^{\infty} f_h z^h$$

is a formal power series, with p -adic algebraic coefficients, which satisfies an algebraic differential equation, then there exist two positive constants γ_1 and γ_2 such that for all $h \geq 0$,

$$|f_h|_p \leq \gamma_1 (h!)^{\gamma_2} ,$$

where $| \cdot |_p$ denotes the p -adic valuation so normalized that $|p|_p = 1/p$.

In order to make the whole proof self-contained, we shall give a detailed proof, parts of which are reproduced from Mahler⁶.

Let K be an algebraic number field generated (over \mathbb{Q}) by all coefficients of f . Let $K[[z]]$ be the ring of formal power series (in z) with coefficients in K . Let $f(z)$ satisfy an algebraic differential equation

$$(1) \quad F(f) := F(z; f, f', f'', \dots, f^{(m)}) = 0 .$$

By a lemma of Ritt and Gourin⁸, we may assume that this algebraic differential equation (1) has rational coefficients. We further assume that (1) is an algebraic differential equation which has lowest order m and which is of lowest degree in $f^{(m)}$ among all such algebraic differential equations of this order m . Since factors of F not involving $f^{(m)}$ may be divided out, without loss of generality, the polynomial $F(z; w_0, w_1, \dots, w_m)$ may be assumed to be irreducible. It is convenient to use the notation

$$F_\mu(z; w_0, w_1, \dots, w_m) = \frac{\partial}{\partial w_\mu} F(z; w_0, w_1, \dots, w_m) \quad (\mu = 0, 1, \dots, m)$$

and

$$F_{(\mu)}(w) := F_\mu(z; w, w', \dots, w^{(m)}) \quad (\mu = 0, 1, \dots, m)$$

where w may be any series in $K[[z]]$. The assumptions about F imply then that

$$(2) \quad F_{(m)}(f) \equiv F_m(z; f, f', \dots, f^{(m)}) \neq 0 .$$

For $F(w)$ depends explicitly on $w^{(m)}$; therefore

$$F_{(m)}(f) = 0$$

would be a differential equation for f of lower order, or of the same order but of lower degree, than the differential equation (1), contrary to the hypothesis.

The differential operator $F(w)$ can be written in the explicit form

$$(3) \quad F(w) = \sum_{(k)} p_{(k)}(z) w^{(k_1)} \dots w^{(k_N)} .$$

Here the summation

$$\sum_{(k)}$$

extends over all ordered system $(k) = (k_1, k_2, \dots, k_N)$ of integers for which

$$(4) \quad 0 \leq k_1 \leq m, \dots, 0 \leq k_N \leq m; \quad k_1 \leq k_2 \leq \dots \leq k_N; \quad 0 \leq N \leq n,$$

where n is a fixed positive integer, and the coefficients $p_{(k)}(z)$ are polynomials in $\mathbb{Z}[z]$. The integer N varies with (k) , and there is exactly one improper system with $N = 0$ which will be defined by (ω) . The corresponding term $p_{(\omega)}(z)$ in (3) has no factors $w^{(j)}$ and is a polynomial in z alone.

We require an explicit expression for the derivatives

$$(5) \quad F^{(h)}(w) = \left(\frac{d}{dz}\right)^h F(w) \quad (h = 1, 2, 3, \dots)$$

of $F(w)$ and shall obtain it by means of the following lemma.

Lemma. Let $h \geq 1$ and $N \geq 0$ be any two integers, and let w_0, w_1, \dots, w_N be $N+1$ arbitrary series in $K[[z]]$. Then

$$(6) \quad \left(\frac{d}{dz}\right)^h (w_0 w_1 \dots w_N) = h! \sum_{\lambda_0, \lambda_1, \dots, \lambda_N} \frac{w_0^{(\lambda_0)}}{\lambda_0!} \cdot \frac{w_1^{(\lambda_1)}}{\lambda_1!} \dots \frac{w_N^{(\lambda_N)}}{\lambda_N!},$$

where the summation extends over all ordered system of integers $\lambda_0, \lambda_1, \dots, \lambda_N$ such that

$$(7) \quad \lambda_0 \geq 0, \lambda_1 \geq 0, \dots, \lambda_N \geq 0; \quad \lambda_0 + \lambda_1 + \dots + \lambda_N = h.$$

Proof. The assertion is evidently true if $h = 1$; assume it has already been established for some order $h \geq 1$. We show that it is then valid also for the order $h+1$ and hence is true for all orders h .

On differentiating (6) once more,

$$\begin{aligned} \left(\frac{d}{dz}\right)^{h+1} (w_0 w_1 \dots w_N) &= h! \sum_{\lambda_0, \lambda_1, \dots, \lambda_N} \sum_{\nu=0}^N \frac{w_0^{(\lambda_0)}}{\lambda_0!} \dots \frac{w_\nu^{(\lambda_\nu+1)}}{\lambda_\nu!} \dots \frac{w_N^{(\lambda_N)}}{\lambda_N!} \\ &= h! \sum_{\mu_0, \mu_1, \dots, \mu_N} \left(\sum_{\nu=0}^N \mu_\nu \right) \frac{w_0^{(\mu_0)}}{\mu_0!} \frac{w_1^{(\mu_1)}}{\mu_1!} \dots \frac{w_N^{(\mu_N)}}{\mu_N!}, \end{aligned}$$

where the new summation extends over all ordered system of integers $\mu_0, \mu_1, \dots, \mu_N$ for which

$$\mu_0 \geq 0, \mu_1 \geq 0, \dots, \mu_N \geq 0; \quad \mu_0 + \mu_1 + \dots + \mu_N = h+1.$$

Now

$$h! \sum_{\nu=0}^N \mu_\nu = (h+1)! ,$$

whence the assertion.

The lemma just proved will now be applied to all the separate terms

$$(8) \quad p_{(k)}(z) w^{(k_1)} \dots w^{(k_N)}$$

in the representation (3) of $F(w)$. It follows immediately from the lemma that

$$(9) \quad F^{(h)}(w) = h! \sum_{(k)} \sum_{[\lambda]} \frac{p_{(k)}^{(\lambda_0)}(z)}{\lambda_0!} \frac{w^{(k_1 + \lambda_1)}}{\lambda_1!} \dots \frac{w^{(k_N + \lambda_N)}}{\lambda_N!} .$$

Here the inner sum

$$\sum_{[\lambda]}$$

is extended over all ordered systems $[\lambda] = [\lambda_0, \lambda_1, \dots, \lambda_N]$ of integers which satisfy the conditions (7), while N has the same value as in the term (8) from which the sum arises. There is in particular exactly one term $p_{(\omega)}^{(h)}(z)$ on the right hand side of (9) for which $N = 0$. This term is independent of w and vanishes as soon as h exceeds the degree of the polynomial $P_{(\omega)}(z)$.

Denote next by j any integer in the interval

$$(10) \quad 0 \leq j \leq \left[\frac{h-1}{2} \right] ,$$

put

$$(11) \quad H = h + m - j ,$$

and denote by

$$(12) \quad F^{(h,H)}(w) w^{(H)}$$

the sum of all terms on the right hand side of (9) which have at least one factor $w^{(H)}$. Further, let

$$(13) \quad F_{(k)}^{(h,H)}(w) w^{(H)}$$

be the sum of all those contributions to $F^{(h,H)}(w) w^{(H)}$ which come from the h th derivative

$$(14) \quad \left(\frac{d}{dz} \right)^h (p_{(k)}(z) w^{(k_1)} \dots w^{(k_N)}) = h! \sum_{[\lambda]} \frac{p_{(k)}^{(\lambda_0)}(z)}{\lambda_0!} \frac{w^{(k_1 + \lambda_1)}}{\lambda_1!} \dots \frac{w^{(k_N + \lambda_N)}}{\lambda_N!}$$

of the term

$$t_{(k)} = p_{(k)}(z) w^{(k_1)} \dots w^{(k_N)}$$

in the representation (3) of $F(w)$. Evidently

$$(15) \quad F^{(h,H)}(w) = \sum_{(k)} F_{(k)}^{(h,H)}(w) ,$$

where trivially

$$(16) \quad F_{(\omega)}^{(h,H)}(w) = 0 .$$

It therefore suffices to consider only those terms $t_{(k)}$ for which $(k) \neq (\omega)$ and hence $1 \leq N \leq n$.

because, by (17), the terms for which ν does not satisfy the conditions (18) are equal to 0 .

The formula can finally be written as

$$(20) \quad F_{(k)}^{(h,H)}(w) = \sum_{\mu=0}^N \binom{h}{H-\mu} \left(\frac{d}{dz}\right)^{h-H+\mu} \frac{\partial}{\partial w^{(\mu)}} (p_{(k)}(z) w^{(k_1)} \dots w^{(k_N)}),$$

since evidently

$$\frac{\partial}{\partial w^{(\mu)}} (p_{(k)}(z) w^{(k_1)} \dots w^{(k_N)}) = \sum_{\nu} \frac{\partial}{\partial w^{(k_\nu)}} (p_{(k)}(z) w^{(k_1)} \dots w^{(k_N)}) ,$$

where ν in \sum runs over all suffixes $1, 2, \dots, N$ for which $k_\nu = \mu$.

On combining the two relations (15) and (20) with the representation (3) of $F(w)$ and making use of the notation $F_{(\mu)}(w)$ introduced earlier, we arrive at the simple formula

$$(21) \quad F_{(k)}^{(h,H)}(w) = \sum_{\mu=0}^m \binom{h}{H-\mu} \left(\frac{d}{dz}\right)^{h-H+\mu} F_{(\mu)}(w) ,$$

which is due to Kakeya.¹

The basic identities (9) and (21) hold for all series w in $K[z]$. In particular, let

$$w = f.$$

Since $F(f) = 0$, we obtain then firstly the equations

$$(22) \quad F^{(h)}(f) = h! \sum_{(k)} \sum_{[\lambda]} \frac{p_{(k)}^{(\lambda_0)}(z)}{\lambda_0!} \frac{f^{(k_1+\lambda_1)}}{\lambda_1!} \dots \frac{f^{(k_N+\lambda_N)}}{\lambda_N!} = 0 \quad (h = 1, 2, \dots) ,$$

and secondly, for all $h = 1, 2, 3, \dots$ and all $j = 0, 1, \dots, [\frac{1}{2}(h-1)]$, the formula

$$(23) \quad F_{(k)}^{(h,H)}(f) = \sum_{\mu=0}^m \binom{h}{H-\mu} \left(\frac{d}{dz}\right)^{h-H+\mu} F_{(\mu)}(f)$$

for the coefficient of $f^{(H)} = f^{(h-m+j)}$ in (22).

In (22) and (23) put $z = 0$. Since

$$p_{(k)}^{(\lambda_0)}(z)|_{z=0} = p_{(k)}^{(\lambda_0)}(0) \quad \text{and} \quad f^{(\ell)}|_{z=0} = \ell! f ,$$

we find that

$$(24) \quad h! \sum_{(k)} \sum_{[\lambda]} \frac{p_{(k)}^{(\lambda_0)}(0)}{\lambda_0!} \frac{(k_1+\lambda_1)!}{\lambda_1!} \dots \frac{(k_N+\lambda_N)!}{\lambda_N!} f_{k_1+\lambda_1} \dots f_{k_N+\lambda_N} = 0$$

(h = 1, 2, 3, ...)

and here the coefficients of $H! f_H$ on the left hand side is given by

$$(25) \quad F_{(k)}^{(h,H)}(f)|_{z=0} = \sum_{\mu=0}^m \binom{h}{H-\mu} \left(\frac{d}{dz}\right)^{h-H+\mu} F_{(\mu)}(f)|_{z=0} .$$

The $m + 1$ expressions $F_{(\mu)}(f)$ are elements of $K[[z]]$; let

$$(26) \quad F_{(\mu)}(f) = \sum_{h=0}^{\infty} F_{\mu,h} z^h \quad (\mu = 0, 1, \dots, m)$$

be their formal power series with coefficients in K . Since from (2)

$$F_{(m)}(f) \neq 0,$$

not all the coefficients $F_{m,h}$ are 0, and there exists an integer $t \geq 0$ such that

$$(27) \quad F_{m,0} = F_{m,1} = \dots = F_{m,t-1} = 0, \text{ but } F_{m,t} \neq 0.$$

From the power series (26)

$$\left(\frac{d}{dz}\right)^{h-H+\mu} F_{(\mu)}(f)|_{z=0} = (h-H+\mu)! F_{\mu,h-H+\mu} \quad (\mu = 0, 1, \dots, m),$$

whence, by (25),

$$F^{(h,H)}(f)|_{z=0} = \sum_{\mu=0}^m \binom{h}{H-\mu} (h-H+\mu)! F_{\mu,h-H+\mu}.$$

Replace here μ by $m - \mu$; since $H = h + m - j$, the formula then takes the simpler form

$$(28) \quad F^{(h,H)}(f)|_{z=0} = \sum_{\mu=0}^m \binom{h}{j-\mu} (j-\mu)! F_{m-\mu,j-\mu}$$

where terms in the sum on the right hand side with $\mu > j$ are put equal to 0. All these expressions evidently are polynomials in h with coefficients in K .

In particular, for $j = t$ and hence $H = h + m - t$, the polynomial

$$F^{(h,H)}(f)|_{z=0} = \binom{h}{t} t! F_{m,t} + \sum_{\mu=1}^m \binom{h}{t-\mu} (t-\mu)! F_{m-\mu,t-\mu}$$

is by (27) of the exact degree t in h and certainly does not vanish identically. Hence there exists a smallest integer s satisfying

$$0 \leq s \leq t$$

such that also the polynomial in h ,

$$(29) \quad F^{(h,H)}(f)|_{z=0}, \quad \text{where } H = h + m - s,$$

is not identically 0, while for $j = 0, 1, \dots, s-1$ the polynomials (28) vanish identically. With this choice of s , H will from now on always be defined by

$$(30) \quad H = h + m - s;$$

hence H differs from h only by a constant integer which may be positive, negative, or zero.

It follows now, firstly, that the left hand side of the equation (24) does not involve any of the coefficients

$$f_{H+1}, f_{H+2}, \dots, f_{H+s},$$

where

$$c_1 = H(H-1) \dots (H-m+1)$$

$$c_2 = H(H-1) \dots (H-m+2)$$

⋮

$$c_m = 1.$$

Let

$$g_0 = \frac{c_1 a_{10}}{b_1} + \dots + \frac{c_m a_{m0}}{b_m}$$

$$g_1 = \frac{c_1 a_{11}}{b_1} + \dots + \frac{c_m a_{m1}}{b_m}$$

⋮

$$g_{d-1} = \frac{c_1 a_{1,d-1}}{b_1} + \dots + \frac{c_m a_{m,d-1}}{b_m} .$$

Let also

$$l := \max (|l_0|, \dots, |l_d|)$$

$$G := B \max (|g_0|, \dots, |g_{d-1}|)$$

$$\leq B \frac{\max_{i,j} |a_{ij}|}{\min_{i,j} |b_i|} m |H(H-1) \dots (H-m+1)| \leq CH^m ,$$

where C is a constant independent of H . Since

$$\text{Res}(M, E) = \left| \begin{array}{cccc} l_0 & \dots & l_d & \\ \vdots & & & \\ & l_0 & \dots & l_d \\ Bg_0 & \dots & Bg_{d-1} & \\ \vdots & & & \\ & Bg_0 & \dots & Bg_{d-1} \end{array} \right| \begin{array}{l} \left. \vphantom{\begin{array}{c} l_0 \\ \vdots \\ l_0 \\ Bg_0 \end{array}} \right\} d-1 \text{ rows} \\ \left. \vphantom{\begin{array}{c} Bg_0 \\ \vdots \\ Bg_0 \end{array}} \right\} d \text{ rows} \end{array}$$

then

$$\begin{aligned} 1 \leq |\text{Res}(M, E)| &\leq (2d-1)! l^{d-1} G^d \\ &\leq (2d-1)! l^{d-1} (CH^m)^d = C_1 H^{C_2} , \end{aligned}$$

where C_1, C_2 are positive constants independent of H . Now, by a well-known result on resultant¹⁰, there exist two polynomials $X(z)$ and $X^*(z)$ with rational integral coefficients such that

$$\begin{aligned} 0 \neq \text{Res}(M, E) &= M(\theta)X(\theta) + E(\theta)X^*(\theta) \\ &= E(\theta)X^*(\theta) . \end{aligned}$$

Since $X^*(\theta) \in \mathbb{Z}\mathbb{Z}[\theta]$ and θ is a p -adic algebraic integer, then

$$|X^*(\theta)|_p \leq 1 \quad .$$

Consequently,

$$\begin{aligned} |E(\theta)|_p &= \left| \frac{\text{Res}(M, E)}{X^*(\theta)} \right|_p \geq |\text{Res}(M, E)|_p \\ &\geq |\text{Res}(M, E)|^{-1} \geq (C_1 H^{C_2})^{-1} \quad , \end{aligned}$$

where we have made use of the fact that $\text{Res}(M, E) \in \mathbb{Z}\mathbb{Z} - \{0\}$. Thus

$$|BF(\theta)|_p \geq (C_1 H^{C_2})^{-1}$$

and so

$$\left| BH! \sum_{\mu=0}^m \frac{F_{m-\mu, s-\mu}}{(h-s+\mu)!} \right| \geq (C_1 H^{C_2})^{-1}$$

i.e.

$$(38) \quad \left| \frac{B(H)}{a(H)} \right|_p \leq \left| H! \sum_{\mu=0}^m \frac{F_{m-\mu, s-\mu}}{(h-s+\mu)!} \right|_p^{-1} \leq C_1 H^{C_2} \quad ,$$

for $H \geq H_0$. Using this last inequality as well as (34) and (35), we get

$$(39) \quad |f_H|_p \leq C_1 H^{C_2} \max_{(k), [\lambda]}^* |f_{k_1+\lambda_1} \cdots f_{k_N+\lambda_N}|_p \quad , H \geq H_0$$

where the maximum is extended over all pairs of systems (k) , $[\lambda]$ for which

$$(40) \quad 1 \leq N \leq n; \quad 0 \leq k_1 + \lambda_1 \leq H-1, \dots, \quad 0 \leq k_N + \lambda_N \leq H-1 \quad .$$

Now define a new ordered system $\{v\} = \{v_1, \dots, v_N\}$ by

$$(42) \quad v_1 = k_1 + \lambda_1, \dots, v_N = k_N + \lambda_N \quad .$$

Then, firstly, by (40),

$$(43) \quad 1 \leq N \leq n; \quad 0 \leq v_1 \leq H-1, \dots, \quad 0 \leq v_N \leq H-1 \quad .$$

Secondly, by the properties of (k) and $[\lambda]$,

$$v_1 + \dots + v_N = (k_1 + \dots + k_N) + h = H + (k_1 + \dots + k_N - m + s)$$

and hence there exists a further positive integer C_3 independent of H such that

$$(44) \quad v_1 + \dots + v_N \leq H + C_3 \quad .$$

We deduce from (39) and the definition of $\{v\}$ that

$$(45) \quad |f_H|_p \leq C_1 H^{C_2} \max_{\{v\}} |f_{v_1} \cdots f_{v_N}|_p \quad \text{for } H \geq H_0 \quad ,$$

where the maximum is extended over all systems $\{v\}$ with the properties (43) and (44). An upper bound for $|f_H|_p$ is now obtained in the following way from the recursive inequality (45).

Assume for simplicity that

$$(46) \quad H_0 > C_3 + 1 \quad .$$

Further choose H_0 numbers $u_0, u_1, \dots, u_{H_0-1}$ such that

$$(47) \quad |f_H|_p \leq \exp(u_H) \quad \text{for } H = 0, \dots, H_0 - 1; \quad 0 < u_0 < u_1 < \dots < u_{H_0-1}$$

and in terms of these define for each suffix $H \geq H_0$ a number u_H by the equation

$$(48) \quad u_H = C_2 \log H + \max_{\{v\} \in S_H} (u_{v_1} + \dots + u_{v_N}) + \log C_1 \quad .$$

Here S_H denotes the set of all ordered system $\{v\} = \{v_1, \dots, v_N\}$ of integers for which (43) and (44) hold, i.e. for which

$$1 \leq N \leq n; \quad 0 \leq v_1 \leq H-1, \dots, 0 \leq v_N \leq H-1; \quad v_1 + \dots + v_N \leq H + C_3 \quad ;$$

S_H is thus the set of all systems $\{v\}$ over which the maximum in (45) is extended.

The definition of u_H implies that

$$(49) \quad |f_H|_p \leq \exp(u_H) \quad \text{for all suffixes } H \geq 0.$$

This is evident for $H \leq H_0 - 1$ and it follows for $H \geq H_0$ by complete induction on H since by (45) and (48)

$$|f_H|_p \leq C_1 \exp(C_2 \log H + \max_{\{v\} \in S_H} (u_{v_1} + \dots + u_{v_N})) = \exp(u_H) \quad .$$

Again suppose $H \geq H_0$, so that by (46),

$$(50) \quad H > C_3 + 1 \quad .$$

By (48),

$$(51) \quad u_{H+1} - u_H = C_2 \log \frac{H+1}{H} + \max_{\{v'\} \in S_{H+1}} (u_{v'_1} + \dots + u_{v'_{N'}}) - \max_{\{v\} \in S_H} (u_{v_1} + \dots + u_{v_N}) \quad .$$

Here, by (43-44), every system in S_H belongs also to S_{H+1} . The maximum over S_{H+1} is therefore not less than that over S_H , and so (51) implies that

$$(52) \quad u_{H+1} - u_H \geq C_2 \log \frac{H+1}{H} > 0 \quad \text{for } H \geq H_0 \quad .$$

On combining this with (47), we see that the u_H form a strictly increasing sequence of positive numbers.

Consider now any system $\{\pi\} = \{\pi_1, \dots, \pi_{N^*}\}$ in S_{H+1} at which the maximum

$$(53) \quad \max_{\{v'\} \in S_{H+1}} (u_{v'_1} + \dots + u_{v'_{N'}}) = u_{\pi_1} + \dots + u_{\pi_{N^*}}$$

is attained. Since the numbers u_H are positive and form an increasing sequence, the suffixes π_1, \dots, π_{N^*} cannot all be zero; and since further

$$\pi_1 + \dots + \pi_{N^*} \leq H + C_3 + 1 \quad \text{and } H > C_3 + 1 \quad ,$$

at most one of these suffixes can be as large as H . Let, say,

$$\pi_{N^*} > 0$$

be the largest of the suffixes π_1, \dots, π_{N^*} (or one of them if several have this same maximum value); then the other suffixes $\pi_1, \dots, \pi_{N^*-1}$ are non-negative and less than H . Hence the system $\{v^0\} = \{v_1^0, \dots, v_{N^*}^0\}$ defined by

$$N^0 = N^*; \quad v_1^0 = \pi_1, \dots, v_{N^*-1}^0 = \pi_{N^*-1}, \quad v_{N^*}^0 = \pi_{N^*} - 1 \geq 0$$

belongs to the set S_H , and hence it follows that

$$\begin{aligned} \max_{\{v\} \in S_H} (u_{v_1} + \dots + u_{v_{N^*}}) &\geq u_{\pi_1} + \dots + u_{\pi_{N^*-1}} + u_{\pi_{N^*} - 1} \\ &= \max_{\{v'\} \in S_{H+1}} (u_{v'_1} + \dots + u_{v'_{N^*}}) - (u_{\pi_{N^*}} - u_{\pi_{N^*} - 1}) \quad . \end{aligned}$$

The formula (51) leads therefore to the inequality

$$(54) \quad u_{H+1} - u_H \leq C_2 \log \frac{H+1}{H} + \max_{\ell=0, \dots, H-1} (u_{\ell+1} - u_\ell) \quad \text{for } H \geq H_0$$

Put

$$v_\ell = u_{\ell+1} - u_\ell \quad \text{and} \quad C_4 = \max(v_0, v_1, \dots, v_{H_0-1}) \quad ,$$

so that C_4 is a positive constant. By (54),

$$v_H \leq C_2 \log \frac{H+1}{H} + \max_{\ell=0, \dots, H-1} v_\ell \quad \text{for } H \geq H_0 \quad ,$$

that is

$$v_H \leq C_2 \log \frac{H+1}{H} + \max(C_4, v_{H_0}, v_{H_0+1}, \dots, v_{H-1}) \quad \text{for } H \geq H_0 \quad .$$

We assert also that

$$(55) \quad v_H \leq C_2 \log \frac{H+1}{H_0} + C_4 \quad \text{for } H \geq H_0 \quad .$$

For this inequality certainly holds if $H = H_0$; assume then that $H > H_0$, and that the inequality has already been proved for all suffixes up to and including $H-1$. Then

$$\max(C_4, v_{H_0}, v_{H_0+1}, \dots, v_{H-1}) \leq C_2 \log \frac{H}{H_0} + C_4 \quad ,$$

whence also

$$v_H \leq C_2 \log \frac{H+1}{H} + (C_2 \log \frac{H}{H_0} + C_4) = C_2 \log \frac{H+1}{H_0} + C_4 \quad ,$$

showing that (55) holds also for the suffix H and therefore is always true.

On putting

$$C_5 = C_4 - C_2 \log H_0$$

(55) can be written in the form

$$u_{H+1} - u_H \leq C_2 \log(H+1) + C_5 \quad \text{for } H \geq H_0 .$$

We apply this inequality successively for the suffixes $H_0, H_0 + 1, \dots, H - 1$ and add all the results, obtaining

$$u_H \leq u_{H_0} + C_2 \log \left(\frac{H!}{H_0!} \right) + C_5(H - H_0) \quad \text{for } H \geq H_0 .$$

By (49), this finally implies that

$$|f_H|_p \leq \left(\frac{H!}{H_0!} \right)^{C_2} \exp(u_{H_0} + C_5(H - H_0)) \quad \text{for } H \geq H_0 .$$

In this formula, $H!$ increases more rapidly than any exponential of H . On replacing H again by h , we have then proved our theorem.

DISCUSSION

Though the method presented above is notationally complex, and the proof is rather lengthy, the principal idea is quite simple. It is actually based upon the explicit relationship (35) which connects one coefficient with those preceding it. It would be of interest to investigate whether the result so derived is best possible, and if not, when it can be improved.

CONCLUSION

We have essentially shown that if a power series, with algebraic coefficients, satisfies an algebraic differential equation, then its coefficients are either zero or are bounded above by explicitly computable quantities.

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